

On the number of empty boxes in the Bernoulli sieve

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Abstract

The Bernoulli sieve is the infinite "balls-in-boxes" occupancy scheme with random frequencies $P_k = W_1 \cdots W_{k-1}(1 - W_k)$, where $(W_k)_{k \in \mathbb{N}}$ are independent copies of a random variable W taking values in $(0, 1)$. Assuming that the number of balls equals n , let L_n denote the number of empty boxes within the occupancy range. The paper proves that, under a regular variation assumption, L_n , properly normalized without centering, weakly converges to a functional of an inverse stable subordinator. Proofs rely upon the observation that $(\log P_k)$ is a perturbed random walk. In particular, some results for general perturbed random walks are derived. The other result of the paper states that whenever L_n weakly converges (without normalization) the limiting law is mixed Poisson.

Keywords: Bernoulli sieve, infinite occupancy scheme, inverse stable subordinator, perturbed random walk

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1 Introduction and main results

The *Bernoulli sieve* is the infinite occupancy scheme with random frequencies

$$P_k := W_1 W_2 \cdots W_{k-1} (1 - W_k), \quad k \in \mathbb{N}, \quad (1)$$

where $(W_k)_{k \in \mathbb{N}}$ are independent copies of a random variable W taking values in $(0, 1)$. This phrase should be interpreted in the sense that the balls are allocated over an infinite array of boxes $1, 2, \dots$ independently conditionally given (P_k) with probability P_j of hitting box j .

Assuming that the number of balls equals n denote by K_n the number of occupied boxes, M_n the index of the last occupied box, and $L_n := M_n - K_n$ the number of empty boxes within the occupancy range. The main purpose of this note is to prove the following theorem which is the first result about the weak convergence of, properly normalized, L_n in the case of divergence $L_n \xrightarrow{P} \infty$, $n \rightarrow \infty$. Unlike all the previous papers about the Bernoulli sieve (see [9] for a survey) lattice laws of $|\log W|$ are allowed.

Theorem 1.1. *Suppose*

$$\mathbb{P}\{|\log W| > x\} \sim x^{-\alpha} \ell_1(x) \quad \text{and} \quad \mathbb{P}\{|\log(1 - W)| > x\} \sim x^{-\beta} \ell_2(x), \quad x \rightarrow \infty,$$

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for some $0 \leq \beta \leq \alpha < 1$ ($\alpha + \beta > 0$) and some ℓ_1 and ℓ_2 slowly varying at ∞ . If $\beta = \alpha > 0$ it is additionally assumed that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}\{|\log W| > x\}}{\mathbb{P}\{|\log(1 - W)| > x\}} = 0.$$

Then

$$\frac{\mathbb{P}\{W \leq 1/n\}}{\mathbb{P}\{1 - W \leq 1/n\}} L_n \xrightarrow{d} \int_0^1 (1 - s)^{-\beta} dX_\alpha^\leftarrow(s) =: Z_{\alpha,\beta}, \quad n \rightarrow \infty,$$

where $(X_\alpha^\leftarrow(t))_{t \geq 0}$ is an inverse α -stable subordinator defined by

$$X_\alpha^\leftarrow(t) := \inf\{s \geq 0 : X_\alpha(s) > t\},$$

where $(X_\alpha(t))_{t \geq 0}$ is an α -stable subordinator with $-\log \mathbb{E}e^{-sX_\alpha(1)} = \Gamma(1 - \alpha)s^\alpha$, $s \geq 0$.

Turning to the case of convergence

$$L_n \xrightarrow{d} L, \quad n \rightarrow \infty, \tag{2}$$

where L is a random variable with proper and nondegenerate probability law, we start by recalling some previously known facts which can also be found in [9].

- Fact 1. Relation (2) holds true whenever $\mathbb{E}|\log W| < \infty$, $\mathbb{E}|\log(1 - W)| < \infty$, and the law of $|\log W|$ is nonlattice. In this case L has the same law as the number of empty boxes in a limiting scheme with infinitely many balls (see [9] for more details).
- Fact 2. If $W \stackrel{d}{=} 1 - W$ then, for every $n \in \mathbb{N}$, L_n has the geometric law starting at zero with success probability $1/2$.
- Fact 3. If W has beta $(\theta, 1)$ law ($\theta > 0$) then L has the mixed Poisson distribution with the parameter distributed like $\theta|\log(1 - W)|$.

As a generalization of the last two facts we prove the following.

Proposition 1.2. *Whenever (2) holds true, L has a mixed Poisson law.*

The number L_n of empty boxes in the Bernoulli sieve can be thought of as the number of zero decrements before the absorption of certain nonincreasing Markov chain starting at n . Proposition 1.2 will be established in Section 5 as a corollary to a more general result formulated in terms of arbitrary nonincreasing Markov chains.

With the exception of Section 5 the rest of the paper is organized as follows. In Section 2 we identify the law of the random variable $Z_{\alpha,\beta}$ defined in Theorem 1.1. Section 3 investigates the weak convergence of a functional defined in terms of general perturbed random walks. Specializing this result to the particular perturbed random walk $(|\log P_k|)$ Theorem 1.1 is then proved in Section 4.

2 Identification of the law of $Z_{\alpha,\beta}$

In this section we will identify the law of the random variable

$$Z_{\alpha,\beta} = \int_0^1 (1 - s)^{-\beta} dX_\alpha^\leftarrow(s)$$

arising in Theorem 1.1. Since the process $(X_\alpha^\leftarrow(t))_{t \geq 0}$ has nondecreasing paths, the integral is interpreted as a pathwise Lebesgue-Stieltjes integral.

Let T be a random variable with the standard exponential law which is independent of $(Y_\alpha(t))_{t \geq 0}$ a drift-free subordinator with no killing and the Lévy measure

$$\nu_\alpha(dt) = \frac{e^{-t/\alpha}}{(1 - e^{-t/\alpha})^{\alpha+1}} 1_{(0,\infty)}(t) dt.$$

One can check that

$$\Phi_\alpha(x) := -\log \mathbb{E} e^{-xY_\alpha(1)} = \frac{\Gamma(1-\alpha)\Gamma(\alpha x + 1)}{\Gamma(\alpha(x-1) + 1)} - 1, \quad x \geq 0.$$

It is known (see, for instance, [13]) that

$$X_\alpha^\leftarrow(1) \stackrel{d}{=} \int_0^T e^{-Y_\alpha(t)} dt.$$

and that $X_\alpha^\leftarrow(1)$ has a Mittag-Leffler law which is uniquely determined by its moments

$$\mathbb{E}(X_\alpha^\leftarrow(1))^n = \frac{n!}{(\Phi_\alpha(1) + 1) \dots (\Phi_\alpha(n) + 1)} = \frac{n!}{\Gamma(1 + n\alpha)\Gamma^n(1 - \alpha)}, \quad n \in \mathbb{N}. \quad (3)$$

According to [16, p. 3245],

$$\mathbb{E} Z_{\alpha,\beta}^n = \frac{n! \alpha^n}{\Gamma^n(1 - \alpha)\Gamma^n(1 + \alpha)} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \prod_{i=1}^n (1 - t_i)^{-\beta} (t_i - t_{i+1})^{\alpha-1} dt_n \dots dt_1,$$

where $t_{n+1} = 0$. Note that our setting requires an additional factor. Changing the order of integration followed by some calculations lead to

$$\begin{aligned} \mathbb{E} Z_{\alpha,\beta}^n &= \frac{n!}{(\Phi_\alpha(c) + 1) \dots (\Phi_\alpha(cn) + 1)} \\ &= \frac{n!}{\prod_{k=1}^n (1 - \alpha + k(\alpha - \beta)) \text{B}(1 - \alpha, 1 + k(\alpha - \beta))}, \quad n \in \mathbb{N}. \end{aligned} \quad (4)$$

where $c := (\alpha - \beta)/\alpha$, which, by [3, Theorem 2(i)], entails the distributional equality

$$Z_{\alpha,\beta} \stackrel{d}{=} \int_0^T e^{-cY_\alpha(t)} dt. \quad (5)$$

From the inequality $\int_0^T e^{-cY_\alpha(t)} dt \leq T$ and the fact that $\mathbb{E} e^{aT} < \infty$, for $a \in (0, 1)$ (or just using the last cited result) we conclude that the law of $Z_{\alpha,\beta}$ has some finite exponential moments and thereby is uniquely determined by its moments.

Since

$$X_\alpha(t) = \inf\{s \geq 0 : L_\alpha(s) > t\},$$

where $(L_\alpha(t))_{t \geq 0}$ is a local time at level 0 for the $2(1 - \alpha)$ -dimensional Bessel process (see [11, p. 555]), we observe that

$$Z_{\alpha,\beta} = \int_0^1 (1 - s)^{-\beta} dL_\alpha(s).$$

Therefore formula (4) can alternatively be obtained from [11, formula (4.3)].

Finally, two particular cases are worth mentioning

$$Z_{\alpha,0} \stackrel{d}{=} X_\alpha^\leftarrow(1) \quad \text{and} \quad Z_{\alpha,\alpha} \stackrel{d}{=} T.$$

3 Results for perturbed random walks

Let $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ be independent copies of a random vector (ξ, η) with $\xi \geq 0$, $\eta \geq 0$ and $\mathbb{P}\{\xi = 0\} < 1$. We make no assumptions about dependence structure of (ξ, η) . Denote by F and G the distribution functions of ξ and η , respectively. Let $(S_n)_{n \in \mathbb{N}_0}$ be a zero-delayed random walk with a step distributed like ξ . A sequence $(T_n)_{n \in \mathbb{N}}$ defined by

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N},$$

will be called a *perturbed random walk*.

Set

$$\rho(t) := \#\{k \in \mathbb{N}_0 : S_k \leq t\} = \inf\{k \in \mathbb{N} : S_k > t\}, \quad t \geq 0,$$

and $U(t) := \mathbb{E}\rho(t)$. Then $U(t)$ is the renewal function of (S_k) . We want to investigate the weak convergence of

$$T(t) := \sum_{k \geq 1} \left(\exp(-te^{-T_k}) - \exp(-te^{-S_{k-1}}) \right), \quad t \geq 0.$$

As it will become clear in Section 4, this problem is relevant to proving Theorem 1.1.

We need two technical results the first of which is an essential improvement over [1, Theorem 1].

Lemma 3.1. *Suppose*

$$1 - F(x) \sim x^{-\alpha} \ell_1(x), \quad x \rightarrow \infty, \quad (6)$$

and let Q be a nonincreasing function such that $Q(0) < \infty$ and

$$Q(x) \sim x^{-\beta} \ell_2(x), \quad x \rightarrow \infty, \quad (7)$$

for some $0 \leq \beta \leq \alpha < 1$ and some ℓ_1 and ℓ_2 slowly varying at ∞ . Then

$$\frac{1 - F(t)}{Q(t)} \int_0^t Q(t-x) \rho(dx) \xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty, \quad (8)$$

along with convergence of expectations

$$\begin{aligned} \frac{1 - F(t)}{Q(t)} \mathbb{E} \int_0^t Q(t-x) \rho(dx) &= \frac{1 - F(t)}{Q(t)} \int_0^t Q(t-x) dU(x) \\ &\rightarrow \mathbb{E} Z_{\alpha, \beta} = \frac{1}{(1-\beta)B(1-\alpha, 1+\alpha-\beta)}. \end{aligned} \quad (9)$$

Proof. It is well-known [4, Theorem 1b] that condition (6) entails

$$(1 - F(t))\rho(t \cdot) \Rightarrow X_{\alpha}^{\leftarrow}(\cdot), \quad t \rightarrow \infty,$$

under M_1 topology in $D[0, 1]$. The one-dimensional convergence holds along with convergence of all moments. In particular,

$$\lim_{t \rightarrow \infty} (1 - F(t))U(t) = \mathbb{E} X_{\alpha}^{\leftarrow}(1) \stackrel{(3)}{=} \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} =: c_{\alpha}. \quad (10)$$

The Skorohod's representation theorem ensures the existence of versions $(\widehat{\rho}(t)) \stackrel{d}{=} (\rho(t))$ and $(\widehat{X}_{\alpha}^{\leftarrow}(t)) \stackrel{d}{=} (X_{\alpha}^{\leftarrow}(t))$ such that

$$(1 - F(t))\widehat{\rho}(t \cdot) \xrightarrow{\text{a.s.}} \widehat{X}_{\alpha}^{\leftarrow}(\cdot), \quad t \rightarrow \infty.$$

Furthermore, we can assume that $(\widehat{\rho}(t))$ is a.s. nondecreasing and that $(\widehat{X}_\alpha^\leftarrow(t))$ is a.s. continuous as it is the case for the original processes. In particular, with probability one,

$$\lim_{t \rightarrow \infty} (1 - F(t))(\widehat{\rho}(t) - \widehat{\rho}(t(1-s))) = \widehat{X}_\alpha^\leftarrow(1) - \widehat{X}_\alpha^\leftarrow(1-s)$$

uniformly on $[0, 1]$, as it is the convergence of monotone functions to a continuous limit. Further, by virtue of (7), the measure μ_t defined by $\mu_t((s, 1]) := \frac{Q(ts)}{Q(t)}$, $s \in [0, 1)$ converges weakly, as $t \rightarrow \infty$, to a measure with density $s \rightarrow \beta s^{-\beta-1}$, $s \in (0, 1)$. Hence, setting $I(t) := \int_0^t Q(t-x)\rho(dx)$, we have, as $t \rightarrow \infty$,

$$\begin{aligned} \frac{1-F(t)}{Q(t)}I(t) &= \int_0^1 \frac{\rho(t) - \rho(t(1-s))}{(1-F(t))^{-1}} \mu_t(ds) + (1-F(t))\rho(t) \\ &\stackrel{d}{=} \int_0^1 \frac{\widehat{\rho}(t) - \widehat{\rho}(t(1-s))}{(1-F(t))^{-1}} \mu_t(ds) + (1-F(t))\widehat{\rho}(t) \\ &\stackrel{\text{a.s.}}{\rightarrow} \int_0^1 (\widehat{X}_\alpha^\leftarrow(1) - \widehat{X}_\alpha^\leftarrow(1-s))\beta s^{-\beta-1} ds + \widehat{X}_\alpha^\leftarrow(1) \\ &= \int_0^1 (1-s)^{-\beta} d\widehat{X}_\alpha^\leftarrow(s) \\ &\stackrel{d}{=} Z_{\alpha,\beta}, \end{aligned} \tag{11}$$

which proves (8).

In view of (10), $\lim_{t \rightarrow \infty} \frac{U(t)-U(t(1-s))}{U(t)} = 1 - (1-s)^\alpha$. Furthermore, the convergence is uniform on $[0, 1]$. Now setting $J(t) := \int_0^t Q(t-x)dU(x)$ and arguing in the same way as above we conclude that, as $t \rightarrow \infty$,

$$\begin{aligned} \frac{1-F(t)}{Q(t)}J(t) &= (1-F(t))U(t) \left(\int_0^1 \frac{U(t) - U(t(1-s))}{U(t)} \mu_t(ds) + 1 \right) \\ &\rightarrow c_\alpha \left(\int_0^1 (1 - (1-s)^\alpha) \beta s^{-\beta-1} ds + 1 \right) \\ &= c_\alpha \int_0^1 (1-s)^{-\beta} \alpha s^{\alpha-1} ds \stackrel{(4)}{=} \mathbb{E}Z_{\alpha,\beta}, \end{aligned}$$

which proves (9). \square

Lemma 3.2. *Let $\Psi(s)$ be the Laplace transform of a random variable θ taking values in $[0, 1]$ with $\mathbb{P}\{\theta > 0\} > 0$. The following functions*

$$\begin{aligned} f_1(t) &:= \Psi(e^t) - \Psi(2e^t), \quad f_2(t) := (1 - \Psi(2e^t))1_{(-\infty, 0)}(t), \\ f_3(t) &:= (1 - \Psi^2(e^t))1_{(-\infty, 0)}(t), \quad f_4(t) := (\Psi(e^t) - \exp(-e^t))1_{(-\infty, 0)}(t), \\ f_5(t) &:= \exp(-e^t)(\Psi(e^t) - \exp(-e^t)), \quad f_6(t) := \exp(-e^t)(1 - \Psi(e^t)) \end{aligned}$$

and

$$f_7(t) := \exp(-e^t) \mathbb{E} \sum_{k \geq 1} \left(\Psi(e^{t-S_k}) - \exp(-e^{t-S_k}) \right)$$

are directly Riemann integrable on \mathbb{R} . In particular,

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}} f_k(t-x)dU(x) = 0, \quad k = 1, 2, 3, 4, 5, 6, 7, \tag{12}$$

whenever $\mathbb{E}\xi = \infty$.

Proof. $k = 1, 2, 3, 4, 5, 6$: The functions $f_k(t)$ are nonnegative and integrable on \mathbb{R} . Indeed, while $\int_{\mathbb{R}} f_1(t)dt = \log 2$, for $k = 2, 3, 4, 5, 6$ the integrability is secured by the finiteness of $\Psi'(0)$. Furthermore, the functions $t \mapsto e^{-t}f_k(t)$ are nonincreasing. It is known that these properties together ensure that the $f_k(t)$ are directly Riemann integrable (see, for instance, the proof of [6, Corollary 2.17]). Finally, an application of the key renewal theorem¹ yields (12).

$k = 7$: The integrability of $f_4(t)$ ensures that $f_7(t)$ is finite. The integrability of $f_7(t)$ on \mathbb{R} is equivalent to the integrability of $t \mapsto f_7(\log t)/t$ on $[0, \infty)$. Now the integrability of the latter function at the neighborhood of 0 follows from the inequality

$$\frac{f_7(\log t)}{t} \leq \frac{\sum_{k \geq 1} \left(\Psi(te^{-S_k}) - \exp(-te^{-S_k}) \right)}{t} \leq \mathbb{E}(1 - \theta) \mathbb{E} \sum_{k \geq 1} e^{-S_k} < \infty,$$

which holds for $t > 0$. The relation $f_7(\log t)/t = o(e^{-t})$, $t \rightarrow \infty$, which follows from (12) for f_4 , ensures the integrability at the neighborhood of $+\infty$. To prove that $t \mapsto e^{-t}f_7(t)$ is nonincreasing it suffices to check that the function $t \mapsto e^{-t} \frac{\Psi(at) - \exp(-at)}{t}$, for any fixed $a > 0$, is nonincreasing. This is easy as the function $t \mapsto e^{-(a\theta+1)t} \frac{1 - e^{-a(1-\theta)t}}{t}$ is completely monotone, hence nonincreasing, and passing to the expectations preserve the monotonicity. An appeal to the key renewal theorem proves (12) for f_7 . \square

Theorem 3.3. *Suppose*

$$1 - F(x) \sim x^{-\alpha} \ell_1(x) \quad \text{and} \quad 1 - G(x) \sim x^{-\beta} \ell_2(x), \quad x \rightarrow \infty, \quad (13)$$

for some $0 \leq \beta \leq \alpha < 1$ and some ℓ_1 and ℓ_2 slowly varying at ∞ . If $\beta = \alpha$ it is additionally assumed that $\lim_{x \rightarrow \infty} \frac{1 - F(x)}{1 - G(x)} = 0$. Then

$$\frac{1 - F(\log t)}{1 - G(\log t)} T(t) \xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty,$$

where the random variable $Z_{\alpha, \beta}$ was defined in Theorem 1.1.

Proof. Set $\varphi(t) := \mathbb{E} \exp(-te^{-\eta})$, $\psi_1(t) := \varphi(2e^t)$, $\psi_2(t) := \varphi^2(e^t)$, $\psi(t) := \psi_1(t) - \psi_2(t)$ and, for $k \in \mathbb{N}$, $\mathcal{F}_k := \sigma((\xi_j, \eta_j) : j \leq k)$, \mathcal{F}_0 being the trivial σ -algebra. We will show that a major part of the variability of $T(t)$ is absorbed by a *renewal shot-noise* process $(V(t))_{t \geq 0}$, where

$$\begin{aligned} V(t) &:= \sum_{k \geq 0} \mathbb{E}(\exp(-te^{-T_{k+1}}) - \exp(-te^{-S_k}) | \mathcal{F}_k) \\ &= \sum_{k \geq 0} (\varphi(te^{-S_k}) - \exp(-te^{-S_k})). \end{aligned}$$

More precisely, we will prove that, as $t \rightarrow \infty$,

$$\frac{1 - F(\log t)}{1 - G(\log t)} \left(T(t) - V(t) \right) \xrightarrow{P} 0 \quad \text{and} \quad \frac{1 - F(\log t)}{1 - G(\log t)} V(t) \xrightarrow{d} Z_{\alpha, \beta}. \quad (14)$$

¹When the law of ξ is d -lattice, the standard form of the key renewal theorem proves (12) with the limit taken over $t \in d\mathbb{N}$. Noting that in the case $\mathbb{E}\xi = \infty$ $\lim_{t \rightarrow \infty} (U(t+y) - U(t)) = 0$, for any $y \in \mathbb{R}$, and using Feller's classical approximation argument (see p. 361-362 in [7]) lead to (12).

It can be checked that

$$q(t) := \mathbb{E} \left(T(t) - V(t) \right)^2 = \sum_{k \geq 0} \left(\varphi(2te^{-S_k}) - \varphi^2(te^{-S_k}) \right).$$

Hence, $q(e^t) = \int_0^\infty \psi(t-z) dU(z)$.

The second condition in (13) is equivalent to

$$\mathbb{P}\{e^{-\eta} \leq y\} \sim (\log(1/y))^{-\beta} \ell_2(\log(1/y)), \quad y \rightarrow +0,$$

hence to

$$\psi_1(t) \sim t^{-\beta} \ell_2(t) \quad \text{and} \quad \psi_2(t) \sim t^{-2\beta} \ell_2^2(t), \quad t \rightarrow \infty,$$

by [5, Theorem 1.7.1]. With this at hand, applying Lemma 3.1 separately to the integrals $\int_0^t \psi_1(t-x) dU(x)$ and $\int_0^t \psi_2(t-x) dU(x)$ yields

$$\int_0^t \psi(t-x) dU(x) \sim \text{const} \frac{1-G(t)}{1-F(t)}, \quad t \rightarrow \infty. \quad (15)$$

Under the present assumptions $\mathbb{E}\xi = \infty$. Therefore, using Lemma 3.2 for f_2 and f_3 gives $\lim_{t \rightarrow \infty} \int_t^\infty \psi(t-x) dU(x) = 0$. Now combining this with (15) leads to

$$q(t) \sim \text{const} \frac{1-G(\log t)}{1-F(\log t)}, \quad t \rightarrow \infty,$$

and the first part of (14) follows by Chebyshev's inequality.

Applying Lemma 3.2 for f_4 allows us to conclude that

$$\lim_{t \rightarrow \infty} \mathbb{E} \sum_{k \geq 0} \left(\varphi(e^{t-S_k}) - \exp(-e^{t-S_k}) \right) 1_{\{S_k > t\}} = 0.$$

Thus the second part of (14) is equivalent to

$$\begin{aligned} & \frac{1-F(t)}{1-G(t)} \sum_{k \geq 0} \left(\varphi(e^{t-S_k}) - \exp(-e^{t-S_k}) \right) 1_{\{S_k \leq t\}} \\ &= \frac{1-F(t)}{1-G(t)} \int_0^t \left(\varphi(e^{t-x}) - \exp(-e^{t-x}) \right) \rho(dx) \\ &\xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty. \end{aligned} \quad (16)$$

Since the function $z(t) := \varphi(e^t) - \mathbb{E}e^{-\eta} \exp(-e^t)$ is nonincreasing and $z(x) \sim 1-G(x) \sim x^{-\beta} \ell_2(x)$, $x \rightarrow \infty$, an application of Lemma 3.1 gives

$$\frac{1-F(t)}{1-G(t)} \int_0^t z(t-x) \rho(dx) \xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty.$$

Similarly,

$$\frac{1-F(t)}{1-G(t)} \int_0^t \varphi(e^{t-x}) \rho(dx) \xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty. \quad (17)$$

Consequently,

$$\frac{1-F(t)}{1-G(t)} \int_0^t \exp(-e^{t-x}) \rho(dx) \xrightarrow{P} 0, \quad t \rightarrow \infty,$$

which together with (17) proves (16) and hence the theorem. \square

Another interesting functional of the perturbed random walk (T_k) is

$$R(t) := \sum_{k=0}^{\infty} 1_{\{S_k \leq t < S_k + \eta_{k+1}\}}, \quad t \geq 0.$$

Note that $(R(t))_{t \geq 0}$ is a shot-noise process which has received some attention in the recent literature. Assuming that ξ and η are independent the process was used to model the number of busy servers in the GI/G/ ∞ queue [14] and the number of active sessions in a computer network [15, 17].

Arguing in a similar but simpler way as in the proof of Theorem 3.3 we can prove the following.

Proposition 3.4. *Under the assumptions of Theorem 3.3,*

$$\frac{1 - F(t)}{1 - G(t)} R(t) \xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty.$$

In a simpler case that ξ and η are independent another proof of this result was given in [17].

4 Proof of Theorem 1.1

Set

$$S_0^* := 0 \quad \text{and} \quad S_k^* := |\log W_1| + \dots + |\log W_k|, \quad k \in \mathbb{N},$$

$$T_k^* := S_{k-1}^* + |\log(1 - W_k)| = \log P_k, \quad k \in \mathbb{N},$$

$$F^*(x) := \mathbb{P}\{|\log W| \leq x\} \quad \text{and} \quad G^*(x) := \mathbb{P}\{|\log(1 - W)| \leq x\},$$

and $\varphi^*(t) := \mathbb{E}e^{-t|\log(1-W)|}$. Since the sequence $(T_k^*)_{k \in \mathbb{N}}$ is a perturbed random walk, the results developed in the previous section can be applied now.

It is clear that the numbers of balls hitting different boxes in the Bernoulli sieve are dependent. Throwing balls at the epochs of a unit rate Poisson process $(\pi_t)_{t \geq 0}$ leads to a familiar simplification. Indeed, denoting by $\pi_t^{(k)}$ the number of points falling in box k we conclude that, conditionally on (P_j) , the $(\pi_t^{(k)})_{t \geq 0}$ are independent Poisson processes with rates P_k 's, and $\pi_t = \sum_{k \geq 1} \pi_t^{(k)}$. The replacement of n by π_t is called *poissonization* and in the present setting it reduces investigating L_n to studying $L(t) := L_{\pi_t}$, where the indices are independent of (L_j) .

In the Bernoulli sieve the variability of the allocation of balls is affected by both randomness in sampling and randomness of frequencies (P_k) . Our first key result states that with respect to the number of empty boxes the sampling variability is negligible in a strong sense whenever the expectation of $|\log W|$ is infinite.

Lemma 4.1. *Whenever $\mathbb{E}|\log W| = \infty$,*

$$L(t) - \mathbb{E}(L(t)|(P_k)) \xrightarrow{P} 0, \quad t \rightarrow \infty.$$

Proof. Using Chebyshev's inequality followed by passing to expectations we conclude that it suffices to prove that

$$\lim_{t \rightarrow \infty} \mathbb{E} \text{Var}(L(t)|(P_k)) = 0. \tag{18}$$

Since $L(t) = \sum_{k \geq 1} 1_{\{\pi_t^{(k)}=0, \sum_{i \geq k+1} \pi_t^{(i)} \geq 1\}}$ we have

$$\begin{aligned}
\text{Var}(L(t)|(P_k)) &= \sum_{k \geq 1} \left(e^{-tP_k} - e^{-2tP_k} \right) \\
&+ \sum_{k \geq 1} e^{-t(1-P_1-\dots-P_{k-1})} \left(2e^{-tP_k} - e^{-t(1-P_1-\dots-P_{k-1})} - 1 \right) \\
&+ 2 \sum_{1 \leq i < j} e^{-t(1-P_1-\dots-P_{i-1})} \left(e^{-tP_j} - e^{-t(1-P_1-\dots-P_{j-1})} \right) \\
&= \sum_{k \geq 1} \left(\exp(-te^{-T_k^*}) - \exp(-2te^{-T_k^*}) \right) \\
&+ \sum_{k \geq 1} \exp(-te^{-S_{k-1}^*}) \left(2\exp(-te^{-T_k^*}) - \exp(-te^{-S_{k-1}^*}) - 1 \right) \\
&+ 2 \sum_{1 \leq i < j} \exp(-te^{-S_{i-1}^*}) \left(\exp(-te^{-T_j^*}) - \exp(-te^{-S_{j-1}^*}) \right) \\
&=: y_1(t) + y_2(t) + 2y_3(t).
\end{aligned}$$

Setting

$$f_1^*(t) := \varphi^*(e^t) - \varphi^*(2e^t), \quad f_5^*(t) := \exp(-e^t)(\varphi^*(e^t) - \exp(-e^t))$$

and

$$f_6^*(t) := \exp(-e^t)(1 - \varphi^*(e^t)), \quad f_7^*(t) := \exp(-e^t) \mathbb{E} \sum_{k \geq 1} \left(\varphi^*(e^{t-S_k^*}) - \exp(-e^{t-S_k^*}) \right),$$

one can check that

$$\mathbb{E}y_1(e^t) = \int_{\mathbb{R}} f_1^*(t-x) dU^*(x), \quad \mathbb{E}y_2(e^t) = \int_{\mathbb{R}} (f_5^*(t-x) - f_6^*(t-x)) dU^*(x)$$

and

$$\mathbb{E}y_3(e^t) = \int_{\mathbb{R}} f_7^*(t-x) dU^*(x).$$

By using (12) for f_1 , f_5 , f_6 and f_7 we conclude that either of these expectations goes to zero, as $t \rightarrow \infty$, thereby proving (18) and hence the lemma. \square

Observe now that

$$\mathbb{E}(L(t)|(P_k)) = \sum_{k \geq 1} \left(e^{-tP_k} - e^{-t(1-P_1-\dots-P_{k-1})} \right) \quad (19)$$

$$= \sum_{k \geq 1} \left(\exp(-te^{-T_k^*}) - \exp(-te^{-S_{k-1}^*}) \right), \quad (20)$$

which is a particular instance of the functional $T(t)$ (see Section 3). Assuming that the assumptions of Theorem 1.1 hold we then conclude, by Theorem 3.3, that, with $a(t) := \frac{1-G^*(\log t)}{1-F^*(\log t)}$,

$$\mathbb{E}(L(t)|(P_k))/a(t) \xrightarrow{d} Z_{\alpha, \beta}, \quad t \rightarrow \infty.$$

An appeal to Lemma 4.1 proves

$$L(t)/a(t) \xrightarrow{d} Z_{\alpha,\beta}, \quad t \rightarrow \infty.$$

It remains to pass from the poissonized occupancy model to the fixed- n model.

For any fixed $\epsilon \in (0, 1)$ and $x > 0$ we have

$$\begin{aligned} \mathbb{P}\{L(t)/a_\epsilon(t) > x\} &\leq \mathbb{P}\{L(t)/a_\epsilon(t) > x, \lfloor (1-\epsilon)t \rfloor \leq \pi_t \leq \lfloor (1+\epsilon)t \rfloor\} + \mathbb{P}\{|\pi_t - t| > \epsilon t\} \\ &\leq \mathbb{P}\left\{\max_{\lfloor (1-\epsilon)t \rfloor \leq i \leq \lfloor (1+\epsilon)t \rfloor} L_i/a_\epsilon(t) > x\right\} + \mathbb{P}\{|\pi_t - t| > \epsilon t\} \\ &= \mathbb{P}\{L_{\lfloor (1-\epsilon)t \rfloor}/a_\epsilon(t) > x\} + \mathbb{P}\{L_{\lfloor (1-\epsilon)t \rfloor} \leq a_\epsilon(t)x, \max_{\lfloor (1-\epsilon)t \rfloor + 1 \leq i \leq \lfloor (1+\epsilon)t \rfloor} L_i > a_\epsilon(t)x\} \\ &\quad + \mathbb{P}\{|\pi_t - t| > \epsilon t\} := I_1(t) + I_2(t) + I_3(t), \end{aligned}$$

where $a_\epsilon(t) := a(\lfloor (1-\epsilon)t \rfloor)$. Similarly,

$$\begin{aligned} \mathbb{P}\{L(t)/\widehat{a}_\epsilon(t) \leq x\} &\leq \mathbb{P}\{L_{\lfloor (1+\epsilon)t \rfloor}/\widehat{a}_\epsilon(t) \leq x\} + \mathbb{P}\{L_{\lfloor (1+\epsilon)t \rfloor} > \widehat{a}_\epsilon(t)x, \min_{\lfloor (1-\epsilon)t \rfloor \leq i \leq \lfloor (1+\epsilon)t \rfloor - 1} L_i \leq \widehat{a}_\epsilon(t)x\} \\ &\quad + \mathbb{P}\{|\pi_t - t| > \epsilon t\} := J_1(t) + J_2(t) + I_3(t), \end{aligned} \tag{21}$$

where $\widehat{a}_\epsilon(t) := a(\lfloor (1+\epsilon)t \rfloor)$.

It is known [10] that frequencies (1) can be considered as the sizes of the component intervals obtained by splitting $[0, 1]$ at points of the multiplicative renewal process $(Q_k)_{k \in \mathbb{N}_0}$, where

$$Q_0 := 1, \quad Q_j := \prod_{i=1}^j W_i, \quad j \in \mathbb{N}.$$

Accordingly, the boxes can be identified with open intervals (Q_k, Q_{k-1}) , and balls with points of an independent sample U_1, \dots, U_n from the uniform distribution on $[0, 1]$ which is independent of (Q_k) . In this representation balls i and j occupy the same box iff points U_i and U_j belong to the same component interval.

If uniform points $U_{\lfloor (1-\epsilon)t \rfloor + 1}, \dots, U_{\lfloor (1+\epsilon)t \rfloor}$ fall to the right from the point $\min_{1 \leq i \leq \lfloor (1-\epsilon)t \rfloor} U_i$ then

$$\max_{\lfloor (1-\epsilon)t \rfloor + 1 \leq i \leq \lfloor (1+\epsilon)t \rfloor} L_i \leq L_{\lfloor (1-\epsilon)t \rfloor} \quad \text{and} \quad L_{\lfloor (1+\epsilon)t \rfloor} \leq \min_{\lfloor (1-\epsilon)t \rfloor \leq i \leq \lfloor (1+\epsilon)t \rfloor - 1} L_i,$$

which means that neither the event defining $I_2(t)$, nor $J_2(t)$ can hold.

Therefore,

$$\begin{aligned} \max(I_2(t), J_2(t)) &\leq \mathbb{P}\left\{\min_{\lfloor (1-\epsilon)t \rfloor + 1 \leq i \leq \lfloor (1+\epsilon)t \rfloor} U_i < \min_{1 \leq i \leq \lfloor (1-\epsilon)t \rfloor} U_i\right\} \\ &= 1 - \frac{\lfloor (1-\epsilon)t \rfloor}{\lfloor (1+\epsilon)t \rfloor}. \end{aligned}$$

By a large deviation result (see, for example, [2]), there exist positive constants δ_1 and δ_2 such that for all $t > 0$

$$I_3(t) \leq \delta_1 e^{-\delta_2 t}.$$

Select now t such that $(1-\epsilon)t = n \in \mathbb{N}$. Then from the calculations above we obtain

$$\mathbb{P}\{L(n/(1-\epsilon))/a(n) > x\} \leq \mathbb{P}\{L_n/a(n) > x\} + 1 - n/[(1+\epsilon)n/(1-\epsilon)] + \delta_1 \exp^{-\delta_2 n/(1-\epsilon)}.$$

Since $a(t)$ is slowly varying, sending first $n \uparrow \infty$ and then $\epsilon \downarrow 0$ we obtain

$$\liminf_{n \rightarrow \infty} \mathbb{P}\{L_n/a(n) > x\} \geq \mathbb{P}\{Z_{\alpha,\beta} > x\}$$

for all $x > 0$ (since the law of $Z_{\alpha,\beta}$ is continuous). The same argument applied to (21) establishes the converse inequality for the upper limit. The proof of the theorem is herewith complete.

5 Number of zero increments of a nonincreasing Markov chain and proof of Proposition 1.2

With $M \in \mathbb{N}_0$ given and any $n \geq M$, $n \in \mathbb{N}$, let $(Y_k(n))_{k \in \mathbb{N}_0}$ be a nonincreasing Markov chain with $Y_0(n) = n$, state space \mathbb{N} and transition probabilities

$$\begin{aligned}\mathbb{P}\{Y_k(n) = j | Y_{k-1}(n) = i\} &= s_{i,j}, \quad i \geq M+1 \text{ and either } M < j \leq i \text{ or } M = j < i, \\ \mathbb{P}\{Y_k(n) = j | Y_{k-1}(n) = i\} &= 0, \quad i < j, \\ \mathbb{P}\{Y_k(n) = M | Y_{k-1}(n) = M\} &= 1.\end{aligned}$$

Denote by

$$Z_n := \#\{k \in \mathbb{N}_0 : Y_k(n) - Y_{k+1}(n) = 0, Y_k(n) > M\}$$

the number of zero decrements of the Markov chain before the absorption. Assuming that, for every $M+1 \leq i \leq n$, $s_{i,i-1} > 0$, the absorption at state M is certain, and Z_n is a.s. finite.

Since L_n is the number of zero decrements before the absorption of a Markov chain with $M = 0$ and

$$s_{i,j} = \binom{i}{j} \mathbb{E} W^j (1-W)^{i-j},$$

Proposition 1.2 is an immediate consequence of the following result.

Proposition 5.1. *If $Z_n \xrightarrow{d} Z$, $n \rightarrow \infty$, where a random variable Z has a proper and nondegenerate probability law then this law is mixed Poisson.*

Proof. While the chain stays in state $j > M$ the contribution to Z_n is made by a random variable R_j which is equal to the number of times the chain stays in j , hence the representation

$$Z_n = \sum_{k \geq 0} R_{\hat{Y}_k(n)} 1_{\{\hat{Y}_k(n) > M\}}, \quad (22)$$

where $(\hat{Y}_k(n))_{k \in \mathbb{N}_0}$ is the corresponding to $(Y_k(n))$ decreasing Markov chain with $\hat{Y}_0(n) = n$ and transition probabilities

$$\hat{s}_{i,j} = \frac{s_{i,j}}{1 - s_{i,i}}, \quad i > j \geq M.$$

It is clear that $(R_j)_{M+1 \leq j \leq n}$ are independent random variables which are independent of $(\hat{Y}_k(n))$, and R_j has the geometric distribution with success probability $1 - s_{j,j}$, i.e.,

$$\mathbb{P}\{R_j = m\} = (1 - s_{j,j}) s_{j,j}^m, \quad m \in \mathbb{N}_0.$$

Let $(\pi_t)_{t \geq 0}$ be a unit rate Poisson process which is independent of an exponentially distributed random variable T with mean $1/\lambda$. Then π_T has the geometric distribution with success probability $\lambda/(\lambda + 1)$. Conditioning in (22) on the chain and using the latter observation along with the independent increments property of Poisson processes lead to the representation

$$Z_n \stackrel{d}{=} \pi^* \left(\sum_{k \geq 0} T_{\hat{Y}_k(n)} 1_{\{\hat{Y}_k(n) > M\}} \right),$$

where $(T_j)_{M+1 \leq j \leq n}$ are independent random variables which are independent of $(\hat{Y}_k(n))$, and T_j has the exponential distribution with mean $s_{j,j}/(1 - s_{j,j})$, and $(\pi^*(t))_{t \geq 0}$ is a unit rate Poisson process which is independent of everything else. Since Z_n converges in distribution, the sequence in the parantheses must converge, too, and the result follows. \square

As a generalization of the Fact 2 stated on p. 2 we will prove the following

Proposition 5.2. *If $s_{j,M} = s_{j,j}$, for every $j \geq M + 1$, then, for every $n \geq M$, Z_n has the geometric law starting at zero with success probability $1/2$.*

Proof. By the assumption, $s_{M+1,M} = s_{M+1,M+1} = 1/2$, and the result for Z_{m+1} follows from (22).

Assume the statement holds for $M + 1 \leq k < n$, i.e., $\mathbb{P}\{Z_k = j\} = 2^{-j-1}$, $j \in \mathbb{N}_0$. Conditioning on $Y_1(n)$ and noting that $Z_M = 0$ we conclude that

$$\mathbb{P}\{Z_n = 0\} = s_{n,M} + \sum_{k=M+1}^{n-1} \mathbb{P}\{Z_k = 0\} s_{n,k} = s_{n,M} + 1/2(1 - s_{n,M} - s_{n,n}) = 1/2.$$

Similarly, for $j \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}\{Z_n = j\} &= \mathbb{P}\{Z_n = j-1\} s_{n,n} + \sum_{k=M+1}^{n-1} \mathbb{P}\{Z_k = j\} s_{n,k} \\ &= \mathbb{P}\{Z_n = j-1\} s_{n,n} + 1/2(1 - s_{n,M} - s_{n,n}). \end{aligned}$$

Using this for $j = 1$ and recalling that $\mathbb{P}\{Z_n = 0\} = 1/2$ gives $\mathbb{P}\{Z_n = 1\} = 1/4$. Repeating this argument for consecutive values of j 's completes the proof. \square

Now we want to apply the results of this section to another nonincreasing Markov chain. Let $(\tau_k)_{k \in \mathbb{N}}$ be independent copies of a random variable τ with distribution

$$p_j := \mathbb{P}\{\tau = j\}, \quad j \in \mathbb{N}, \quad p_1 > 0.$$

The *random walk with barrier* $n \in \mathbb{N}$ (see [13] for more details) is a sequence $(W_k(n))_{k \in \mathbb{N}_0}$ defined as follows:

$$W_0(n) := 0 \quad \text{and} \quad W_k(n) := W_{k-1}(n) + \tau_k 1_{\{W_{k-1}(n) + \tau_k < n\}}, \quad k \in \mathbb{N}.$$

Plainly, $(n - W_k(n))_{k \in \mathbb{N}_0}$ is a nonincreasing Markov chain with

$$s_{i,j} = p_{i-j}, \quad i > j, \quad \text{and} \quad s_{i,i} = \sum_{k \geq i} p_k,$$

which starts at n and eventually get absorbed in the state 1. Let Z_n^* be the number of zero decrements before the absorption of this chain. It was shown in [12, Theorem 1.1] that, provided $\mathbb{E}\tau < \infty$, Z_n^* converges in distribution. By the virtue of Proposition 5.1, this can be complemented by the statement that the limiting law is mixed Poisson. Finally, using Proposition 5.2 we conclude that, for every $n \geq 2$,

$$\mathbb{P}\{Z_n^* = m\} = 2^{-m-1}, \quad m \in \mathbb{N}_0,$$

provided $p_j = 2^{-j}$, $j \in \mathbb{N}$.

References

- [1] ANDERSON, K. K. AND ATHREYA, K. B. (1987). A renewal theorem in the infinite mean case. *Ann. Probab.* **15**, 388–393.
- [2] BAHADUR, R.R. (1971). Some limit theorems in statistics. CBMS Regional conference series in applied mathematics. **4**. Philadelphia: SIAM.
- [3] BERTOIN, J. AND YOR, M. (2005). Exponential functionals of Lévy processes. *Probability Surveys*. **2**, 191–212.
- [4] BINGHAM, N. H. (1973). Maxima of sums of random variables and suprema of stable processes. *Z. Wahrsch. verw. Gebiete*. **26**, 273–296.
- [5] BINGHAM N. H., GOLDIE C. M., AND TEUGELS, J. L. (1989). Regular variation. Cambridge: Cambridge University Press.
- [6] DURRETT, R. AND LIGGETT, T. M. (1983). Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete*. **64**, 275–301.
- [7] FELLER, W. (1971). An introduction to probability theory and its applications, Vol. 2, 2nd edition, John Wiley & Sons, New York etc.
- [8] GNEDIN, A., IKSANOV, A. AND MARYNYCH, A. (2010). Limit theorems for the number of occupied boxes in the Bernoulli sieve. *Theory of Stochastic Processes*. **16(32)**, 44–57.
- [9] GNEDIN, A., IKSANOV, A. AND MARYNYCH, A. (2010). The Bernoulli sieve: an overview. *Discr. Math. Theoret. Comput. Sci. Proceedings Series*, **AM**, 329–342.
- [10] GNEDIN, A., IKSANOV, A., NEGADAJLOV, P., AND ROESLER, U. (2009). The Bernoulli sieve revisited. *Ann. Appl. Prob.* **19**, 1634–1655.
- [11] GRADINARU, M., ROYNETTE, B., VALLOIS, P. AND YOR, M. (1999). Abel transform and integrals of Bessel local times. *Ann. Inst. Henri Poincaré*. **35**, 531–572.
- [12] IKSANOV, A. AND NEGADAJLOV, P. (2008). On the number of zero increments of random walks with a barrier. *Discrete Mathematics and Theoretical Computer Science*, Proceedings Series. **AG**, 247–254.
- [13] IKSANOV, A. AND MÖHLE, M. (2008). On the number of jumps of random walks with a barrier. *Adv. Appl. Probab.* **40**, 206–228.
- [14] KAPLAN, N. (1975). Limit theorems for a $GI/G/\infty$ queue. *Ann. Probab.* **3**, 780–789.
- [15] KONSTANTOPOULOS, T. AND LIN, S. (1998). Macroscopic models for long-range dependent network traffic. *Queueing Systems*. **28**, 215–243.
- [16] MAGDZIARZ, M. (2009). Stochastic representation of subdiffusion processes with time-dependent drift. *Stoch. Proc. Appl.* **119**, 3238–3252.
- [17] MIKOSCH, T. AND RESNICK, S. (2006). Activity rates with very heavy tails. *Stoch. Proc. Appl.* **116**, 131–155.